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On Invariant Measures of the 2D Euler Equation

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In this article we study the class of the microcanonical invariant measures for the 2D Euler equation under periodic boundary conditions and show that these measures are different from those that are the limits of the stationary measures for randomly forced 2D Navier-Stokes equation as the viscosity tends to zero.

KEY WORDS: 2D Euler equation; 2D Navier-Stokes equation; 2D turbulence; microcanonical measures.

1. INTRODUCTION

Despite the fact (or maybe because of it) that every one has some intuition about turbulence, the strict definition of the notion of turbulence is rather controversial. One of the most popular mathematical points of view is that "turbulence happens" because of certain specific properties of the invariant measure(s) of the corresponding PDE (more precisely the stochastic version of the PDE). These "specific properties" are yet to be precisely formulated and very little is known in this direction. Indeed, questions of existence and uniqueness of stationary measures can be difficult.

Despite being notoriously difficult, this approach allows us to speak about turbulence for virtually any PDE. However, the notion of turbulence with regard to the Euler and the Navier-Stokes equations is more natural from the physical point of view. Within the last few years a number of papers concerning with 2D statistical hydrodynamics have appeared. In our language, they studied invariant/stationary measures for 2D Euler and Navier-Stokes equations. Invariant measure for the deterministic Euler equation are not unique (because there are conserved quantities). However, physically relevant are only those that "comes"

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from the Navier-Stokes equation, i.e., the limiting measures for the Navier-Stokes equation when the viscosity tends to zero.

Hence, there is the first question (Q): *among all invariant measures for the Euler equation can we describe those that can possibly be the limiting stationary measures of Navier-Stokes equations, and what is the limiting procedure*? First, we explain the second part of this question giving the answer for the 2D case due to Kuksin [6].

We remind the reader that an invariant measure is a measure on the phase function space H which is preserved by the flow map of the (deterministic) PDE in question. That is, if Φ_t is the flow map for the PDE, we define the (push forward) semigroup $\Phi_t^\star$ acting on the space $M(H)$ of all bounded measures on $(H, \mathcal{B}(H))$ by

$$\Phi_t^\star(\mu)(\Gamma) = \mu((\Phi_t^\star)^{-1}(\Gamma)) \qquad \Gamma \in \mathcal{B}(H), \mu \in M(H).$$

Here $\mathcal{B}(H)$ denotes the Borel σ-algebra. Let $M_1^+(H)$ be the subset of $M(H)$, consisting of all probability measures. For the stochastic case the flow Φ_t is not defined, however the semigroup $\Phi_t^\star$ is defined naturally on $M_1^+(H)$, see, e.g., [2, chpt 11]. From now on we consider only probability measures. A measure $\mu \in M_1^+(H)$ is said to be *stationary* (*invariant*) for PDE in question if $\Phi_t^\star\mu \equiv \mu \quad \forall t > 0$.

Consider the small viscosity periodic 2D Navier-Stokes (NS) system, perturbed by a small random force:

$$\dot{\boldsymbol{u}} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \nu^{\varkappa}\boldsymbol{\eta}(t,\boldsymbol{x}), \quad 0 < \nu \le 1, \tag{1}$$

$$\operatorname{div}\boldsymbol{u} = 0, \quad \boldsymbol{u} = \boldsymbol{u}(t,\boldsymbol{x}), \quad p = p(t,\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2).$$

It is assumed that $\int \boldsymbol{u} d\boldsymbol{x} \equiv \int \boldsymbol{\eta} d\boldsymbol{x} \equiv 0$. The force $\boldsymbol{\eta}$ is white in time and smooth in space. Let us fix some positive $\varkappa$. Under some assumptions on force η there exists a unique stationary measure μ_ν for equation (1). What the value of $\varkappa$ should be, to guarantee non-trivial limit(s) of μ_ν as $\nu \to 0$? Kuksin's answer is: $\varkappa = 1/2$ only. If $\varkappa > 1/2$ then the measure μ_ν converges to the delta measure at zero. If $\varkappa < 1/2$ then the support of the measure μ_ν spreads to infinity, as $\nu \to 0$.

The question (Q) makes sense also for the case of Galerkin approximations, and the answer for the second part is the same, i.e., the procedure is the same. (see proofs in [6]). In this note we discuss the first part of the question (Q) for the case of the Galerkin approximations.

Before going to the finite dimension approximations we make some general settings.

We set $\varkappa = 1/2$ for the remainder of this article. To specify assumptions on the random force η we introduce the space

$$\mathcal{H} = \left\{\boldsymbol{u} \in L_2(\mathbb{T}^2;\mathbb{R}^2) : \operatorname{div}\boldsymbol{u} = 0, \int \boldsymbol{u}\, d\boldsymbol{x} = 0\right\}.$$

Let $e_{\boldsymbol{s}}$, $\boldsymbol{s} \in \mathbb{Z}^2 \setminus \{0\}$, be the trigonometric basis in $\mathcal{H}$:

$$e_{\boldsymbol{s}}(\boldsymbol{x}) = \frac{1}{\sqrt{2}\pi} \frac{\boldsymbol{s}^{\perp}}{|\boldsymbol{s}|} \cos(\boldsymbol{s} \cdot \boldsymbol{x}), \quad e_{-\boldsymbol{s}}(\boldsymbol{x}) = \frac{1}{\sqrt{2}\pi} \frac{\boldsymbol{s}^{\perp}}{|\boldsymbol{s}|} \sin(\boldsymbol{s} \cdot \boldsymbol{x}) \quad \text{for } \boldsymbol{s} \in \mathbb{Z}^2_+ .$$

Here $\mathbb{Z}^2_+ = \{\binom{s_1}{s_2} : s_1 > 0 \text{ or } (s_1 = 0 \text{ and } s_2 > 0)\}$, $|\boldsymbol{s}|$ stands for the Euclidean norm of $\boldsymbol{s}$, and $\binom{s_1}{s_2}^{\perp} = \binom{-s_2}{s_1}$.

We assume that

$$\eta = \sum_{\boldsymbol{s} \in \mathbb{Z}^2 \setminus \{0\}} b_{\boldsymbol{s}} \frac{d}{dt} \{\beta_{\boldsymbol{s}}(t)\} e_{\boldsymbol{s}}(x), \tag{2}$$

where $\{b_{\boldsymbol{s}}\}$ are real constants, $\{\beta_{\boldsymbol{s}}(t)\}$ are independent standard Wiener processes, so $\frac{d}{dt}\{\beta_{\boldsymbol{s}}(t)\}$ are the standard independent white noises. We need the assumptions

$$B_0 := \sum_{\boldsymbol{s} \in \mathbb{Z}^2 \setminus \{0\}} b_{\boldsymbol{s}}^2 < \infty \qquad \text{and} \qquad B_1 := \sum_{\boldsymbol{s} \in \mathbb{Z}^2 \setminus \{0\}} |\boldsymbol{s}|^2 b_{\boldsymbol{s}}^2 < \infty. \tag{3}$$

We assume that the real constants $\{b_{\boldsymbol{s}}\}$ satisfying (3) are fixed throughout the article. For the existence of a stationary measure μ_ν on $\mathcal{H}$ for equation (1) see [10].

The measure μ_ν is not necessary unique. The question of its uniqueness is delicate and very interesting but falls outside the scope of the present paper. The questions of uniqueness in various settings are discussed in [3,4,8,9] see also reviews [1, 5] and references therein.

We are interested in all possible weak limit measures (as $\nu \to 0$), i.e., weak limits of all possible sequences μ_{ν_j} as $\nu_j \to 0$.

In [6] it is shown that any sequence of stationary measures $\mu_{\tilde{\nu}_j}$ with $\tilde{\nu}_j \to 0$ contains a weakly convergent subsequence. I.e., there exists a subsequence $\nu_j \to 0$, the corresponding stationary measures μ_{ν_j} and a measure μ_0 such that for any $f \in C_b(\mathcal{H})$, i.e., any continuous and bounded function $f : \mathcal{H} \to \mathbb{R}$, we have

$$\int_{\mathcal{H}} f d\mu_{\nu_j} \to \int_{\mathcal{H}} f d\mu_0.$$

The measure μ_0 is invariant for the (deterministic) Euler equation.

Let $\mathcal{H}_{(N)} = \operatorname{span}\{e_{\boldsymbol{s}} : 0 < |\boldsymbol{s}| \le N\}$ and $P_N : L_2(\mathbb{T}^2; \mathbb{R}^2) \to \mathcal{H}_{(N)}$ be the L^2-orthogonal projector. Let

$$n(N) = \frac{1}{2} \#\{0 < |\boldsymbol{s}| \le N\} \tag{4}$$

and therefore $\dim \mathcal{H}_{(N)} = 2n$. Consider the Galerkin approximation for the Navier Stokes equation

$$\dot{u} - \nu \Delta u + P_N((u \cdot \nabla)u) = \sqrt{\nu}\eta(t, x) \tag{NS_N}$$

and for the Euler equation

$$\dot{u} + P_N((u \cdot \nabla)u) = 0. \tag{E_N}$$

In view of (2) we assume $\eta(t) = \sum_{0<|\boldsymbol{s}|\le N} b_{\boldsymbol{s}} \frac{d}{dt}\beta_{\boldsymbol{s}}(t)e_{\boldsymbol{s}}(\boldsymbol{x})$. The same arguments as apply to (1) imply the existence of a stationary measure μ_ν^N for equation (NS_N).

Theorem 1. *Any sequence of stationary measures $\mu_{\tilde{\nu}_j}^N$ with $\tilde{\nu}_j \to 0$ contains a weakly convergent subsequence.*

Moreover, every limiting measure μ_0^N is an invariant measure for (deterministic) equation (E_N).

Lemma 2. *Any limiting measure μ_0^N constructed in Theorem 1 satisfies*

$$\int_{\mathcal{H}_{(N)}} \|u\|_2^2\, \mu_0^N(du) \le \frac{1}{2} \sum_{0<|\boldsymbol{s}|\le N} |\boldsymbol{s}|^2 b_{\boldsymbol{s}}^2. \tag{5}$$

Here $\|u\|_2$ denotes the H^2-Sobolev norm of u. We recall that for $u = \sum \hat{u}_{\boldsymbol{s}} e_{\boldsymbol{s}}$ the H^k-Sobolev norm $\|u\|_k$ is defined by

$$\|u\|_k^2 = \sum |\hat{u}_{\boldsymbol{s}}|^2 |\boldsymbol{s}|^{2k}.$$

In the sequel we will abbreviate $\hat{u}_{\boldsymbol{s}}$ to $u_{\boldsymbol{s}}$. The proofs of Theorem 1 and Lemma 2 follows by the literal repetition of the proofs of Theorem 3.1 and Lemma 3.2 from [6].

Corollary 3. *Assume (3). Then we have*

$$\int_{\mathcal{H}_{(N)}} \|u\|_2^2\, \mu_0^N(du) \le \frac{1}{2} B_1. \tag{5}$$

Lemma 3. *The Lebesgue measure l^{2n} on $\mathcal{H}_{(N)} \approx \mathbb{R}^{2n}$ is invariant for (E_N).*

Proof: The system (E_N) can be written as

$$\frac{d}{dt} u_{\boldsymbol{s}} = f_{\boldsymbol{s}}(\{u_{\boldsymbol{k}}\}_{0<|\boldsymbol{k}|\le N}),$$

where $u(t,x) = \sum_{0<|\boldsymbol{s}|\le N} u_{\boldsymbol{s}}(t) e_{\boldsymbol{s}}(x)$ and, similarly, $f_{\boldsymbol{s}}$ are defined by

$$P_N((u \cdot \nabla)u) = \sum_{0<|\boldsymbol{s}|\le N} f_{\boldsymbol{s}} e_{\boldsymbol{s}}.$$

To conclude the proof we note that for each $\boldsymbol{s}$ the function $f_{\boldsymbol{s}}$ does not depend on $u_{\boldsymbol{s}}$ and hence $\mathrm{div}_{\{u\}}\{f\} = \sum_{0<|\boldsymbol{s}|\le N} \frac{\partial f_{\boldsymbol{s}}}{\partial u_{\boldsymbol{s}}} = 0$. □

The system (E_N) has two integrals of motion,

$$\text{the energy } E(u) = |u|^2 \quad \text{and the enstrophy } \Omega(u) = \|u\|_1^2.$$

Let $c_1 < c_2$ be two positive real numbers. Define the microcanonical measure $\ell^N_{c_1,c_2}$ to be

$$\ell^N_{c_1,c_2}(du) = M^N_{c_1,c_2}\delta_{E=c_1,\Omega=c_2} l^{2n}(du).$$

The normalizing factor M_{c_1,c_2} is chosen to satisfy $\ell^N_{c_1,c_2}(\mathcal{H}_{(N)}) = 1$. The density $\delta_{E=c_1,\Omega=c_2}$ is a distribution obtained by the pull back procedure from δ_{c_1,c_2} under the mapping

$$\{E, \Omega\} : \mathcal{H}_{(N)} \to \mathbb{R}^2.$$

The pull back for the distributions is defined by approximation by smooth functions. Consider a change of variables $u = u(y_1, y_2, z)$, where $y_1 = E(u) - c_1$, $y_2 = \Omega(u) - c_2$ and z is a $(2n-2)$-dimensional coordinate such that dz maps to the $(2n-2)$-dimensional Lebesgue measure $d\sigma$ on the surface $\{E = c_1, \Omega = c_2\}$ under this change of variables. We note that the surface $\{E = c_1, \Omega = c_2\}$ can also be specified in a parametric form as $\{u(0, 0, z)\}$ where z is the $(2n-2)$ dimensional parameter. We have

$$\begin{aligned}
\langle \delta_{E=c_1,\Omega=c_2}, \varphi(\cdot)\rangle &= \int_{\{u\}} \delta(E(u) - c_1, \Omega(u) - c_2)\varphi(u)du \\
&= \int_{\{z\}}\int_{\{y\}} \delta(y_1, y_2)\varphi(u(y_1, y_2, z))\frac{\partial u}{\partial(y, z)}dydz
\end{aligned}$$

$$= \int_{\{z\}} \varphi(u(0, 0, z))\frac{\partial u}{\partial(y, z)}\bigg|_{y=(0,0)} dz = \int_{\{u\}\cap\{E=c_1\}\cap\{\Omega=c_2\}} \varphi(u)\frac{d\sigma}{[\nabla E, \nabla\Omega]}.$$

Here $[\nabla E, \nabla\Omega]$ is the (2-dimensional) area of the parallelogram constructed by the vectors ∇E and $\nabla\Omega$.

Equivalently, for any open set S the measure $\delta_{E=c_1,\Omega=c_2} l^{2n}(S)$ is defined by

$$\delta_{E=c_1,\Omega=c_2} l^{2n}(S) = \lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{1}{4\varepsilon_1\varepsilon_2} l^{2n}(S \cap \{|E - c_1| < \varepsilon_1\} \cap \{|\Omega - c_2| < \varepsilon_2\}). \tag{6}$$

This measure is *not* proportional to the $(2n-2)$-dimensional Lebesgue measure $d\sigma$ on the surface $\{E = c_1, \Omega = c_2\}$:

$$\delta_{E=c_1,\Omega=c_2} l^{2n}(S) = \int_{S\cap\{E=c_1\}\cap\{\Omega=c_2\}} \frac{d\sigma}{[\nabla E, \nabla\Omega]}.$$

Obviously, for any $c_1 < c_2$ the microcanonical measure $\ell^N_{c_1,c_2}$ is invariant for (E_N).

Hence, we have a family of invariant measures for the equation (E_N)

$$\tilde{\mu}_\varrho(\cdot) = \iint_{\mathbb{R}^2_{++}} \ell^N_{c_1,c_2}(\cdot)\, \varrho(dc_1 dc_2),$$

parameterized by the measures ϱ on $\mathbb{R}^2_{++} = \{c_1, c_2 : 0 < c_1 < c_2\}$.

Physical theories of the $2D$ turbulence often come out with an opinion, which in our terms states that physically relevant invariant measures for the equation (E_N) can be represented as a combination of the microcanonical measures (e.g., see [7]). A possible understanding of what is *physically relevant* is that the measure comes by a limiting procedure from the Navier-Stokes equation. This opinion is motivated by the fact that in reality we deal with very small but positive viscosity. For the $2D$ Navier-Stokes equation, supplemented by the periodic boundary conditions, Kuksin proves (see [6]) that among a wide class of limiting procedures, the only possible physical way is to use "square root of the viscosity scaling" for the force and apply theorem 1. Due to this result we reformulate the corresponding conjecture as follows:

Conjecture 5. *There exist an N-independent measure ϱ on $\mathbb{R}^2_{++}$ such that for all $N \gg 1$ there exists a measure μ^N_0, provided by Theorem 1, that can be represented in the form*

$$\mu^N_0(\cdot) = \iint_{\mathbb{R}^2_{++}} \ell^N_{c_1,c_2}(\cdot)\, \varrho(dc_1 dc_2).$$

Our goal is to show that this conjecture is wrong. This follows from the following theorem.

Theorem 6. *For any positive $c_1 < c_2$ we have*

$$\int_{\mathcal{H}_{(N)}} \|u\|_2^2\, \ell^N_{c_1,c_2}(du) \to \infty \quad \textit{as } N \to \infty. \tag{7}$$

If the conjecture 5 is true, then (7) contradicts to (5′).

We now outline the proof of Theorem 6. First we note that

$$\int_{\mathcal{H}_{(N)}} \|u\|_2^2\; \ell^N_{c_1,c_2}(du) = \sum_{0<|\boldsymbol{s}|\le N} \int_{\mathcal{H}_{(N)}} |\boldsymbol{s}|^4\, |u_{\boldsymbol{s}}|^2\; \ell^N_{c_1,c_2}(du).$$

Then we will show that (see (11) and Proposition 6)

$$\int_{\mathcal{H}_{(N)}} (|u_{\boldsymbol{s}}|^2 + |u_{-\boldsymbol{s}}|^2)\ell^N_{c_1,c_2}(du) \approx (c_2 - c_1)\frac{1}{|\boldsymbol{s}|^2}\frac{1}{n(N)}. \tag{8}$$

We recall (cf. (4)) that $n(N)$ is the cardinality of the intersection $\mathbb{Z}_+^2$ and the closed disk of the radius N, centered at the origin. Using the approximation $n(N) \approx \frac{1}{2}\pi N^2$ and noting that $\sum_{|s|\le|N|} |s|^2 \approx \frac{\pi}{2} N^4$ we obtain

$$\int_{\mathcal{H}_{(N)}} \|u\|_2^2 \, \ell_{c_1,c_2}^N(du) \approx (c_2 - c_1)N^2.$$

This proves (7). Of course the main point is to show (8). The special change of variables reduces the last problem to a pure geometrical problem of the sections of multidimension tetrahedrons (simplexes). Our main technical result, theorem 10, calculates the left hand side of (8) explicitly. We believe that this theorem is of independent interest.

2. TRANSFORMATION OF THE PHASE SPACE

In this section we introduce a change of variables that reduces Theorem 6 to a pure geometrical problem.

Let $\mathbb{Z}_{+,N}^2 = \{\boldsymbol{s} \in \mathbb{Z}_+^2 : 0 < |\boldsymbol{s}| \le N\}$. We recall that the cardinality $\#\{\mathbb{Z}_{+,N}^2\} = n(N)$. Let $\mathcal{H}_{(N)}^+ = \{\{y_{\boldsymbol{s}}\}_{\boldsymbol{s}\in\mathbb{Z}_{+,N}^2} : y_{\boldsymbol{s}} \ge 0\}$. Define a map $Y : \mathcal{H}_{(N)} \to \mathcal{H}_{(N)}^+$ by

$$y_{\boldsymbol{s}} = u_{-\boldsymbol{s}}^2 + u_{\boldsymbol{s}}^2.$$

This map preserves the Lebesgue measure in the following sense. Let $U \subset \mathcal{H}_{(N)}^+$ be any measurable set, then

$$l^n(U) = \pi^{-n} l^{2n}(Y^{-1}(U)). \tag{9}$$

Here n abbreviates $n(N)$; ℓ^n and ℓ^{2n} denote the n-dimensional Lebesgue measure on $\mathcal{H}_{(N)}^+$ and the $2n$-dimensional Lebesgue measure on $\mathcal{H}_{(N)}$, respectively. To prove (9) we note that the map Y is the n^{th} tensor power of the map $Y_1 : \mathbb{R}^2 \to \mathbb{R}_+$, $Y_1(a, b) = a^2 + b^2$, and for the latter the property

$$\ell^1(U) = \pi^{-1}\ell^2(Y_1^{-1}(U))$$

is straightforward.

Hence (in view of (6)), the microcanonical measure ℓ_{c_1,c_2} goes (up to a normalizing factor) into the 2-codimensional Lebesgue $\tilde{\ell}_{c_1,c_2}$ measure of surface

$$\sum_{\boldsymbol{s}\in\mathbb{Z}_{+,N}^2} y_{\boldsymbol{s}} = c_1, \qquad \sum_{\boldsymbol{s}\in\mathbb{Z}_{+,N}^2} |\boldsymbol{s}|^2 y_{\boldsymbol{s}} = c_2. \tag{10}$$

Now we can rewrite the left hand side of (8) as follows

$$\int_{\mathcal{H}_{(N)}} (|u_{\boldsymbol{s}}|^2 + |u_{-\boldsymbol{s}}|^2)\ell_{c_1,c_2}^N(du) = \int_{\mathcal{H}_{(N)}^+} y_{\boldsymbol{s}} \tilde{\ell}_{c_1,c_2}^N(d\boldsymbol{y}) = \int_0^\infty x\rho_{c_1,c_2,N,\boldsymbol{s}}(x)\,dx. \tag{11}$$

Here $\rho_{c_1,c_2,N,\boldsymbol{s}}(x)$ is the (marginal) density of distribution of the $\boldsymbol{s}$-th component $y_{\boldsymbol{s}}$ where the random variable $\{y_{\tilde{\boldsymbol{s}}}\}_{\tilde{\boldsymbol{s}}\in\mathbb{Z}^2_{+,N}}$ are uniformly distributed in polyhedron (10). Our next goal is to find the density $\rho_{c_1,c_2,N,\boldsymbol{s}}(x)$ and its first moment. To conclude this short section we observe the following homogeneity properties:

$$\rho_{c_1,c_2,N,\boldsymbol{s}}(x) = c_1^{-1}\rho_{1,c_2/c_1,N,\boldsymbol{s}}(x/c_1) \tag{12}$$

and

$$\int x\rho_{c_1,c_2,N,\boldsymbol{s}}(x)dx = c_1\int \rho_{1,c_2/c_1,N,\boldsymbol{s}}(x)dx. \tag{13}$$

3. MEASURES OF THE SECTIONS OF THE SIMPLEXES

Let $p_1, p_2, \ldots$ be an arbitrary sequence of real numbers. Fix $n \geq 2$. Assume that $\mathbf{1}^n = (1,\ldots,1) \nparallel \boldsymbol{p}^n = (p_1,\ldots,p_n)$. Consider the following polyhedron in $\mathbb{R}^n$:

$$\begin{cases} x_1 + \cdots + x_n \leq A, \\ p_1x_1 + \cdots + p_nx_n \leq B, \\ x_k \geq 0, \qquad \text{for all } k. \end{cases}$$

By $V_{n,\boldsymbol{p}}(A,B)$ denote its n-dimensional Lebesgue volume. It is assumed that $A > 0$. The goal of this section is the studying the function

$$F_{n,\boldsymbol{p}}(A,B) = \frac{\partial^2}{\partial A\partial B}V_{n,\boldsymbol{p}}(A,B).$$

We note that if $B \geq A\max\{0,p_1,\ldots,p_n\}$ then we have

$$V_{n,\boldsymbol{p}}(A,B) = \mathrm{Vol}\{\boldsymbol{x}\in\mathbb{R}^n : x_k \geq 0, \sum x_k \leq A\} = \frac{1}{n!}A^n,$$

Similarly, if $p_k > 0$ for $k = 1\ldots n$, then for $0 < B \leq A\min\limits_{k=1,\ldots,n} p_k$ we have

$$V_{n,\boldsymbol{p}}(A,B) = \mathrm{Vol}\{\boldsymbol{x}\in\mathbb{R}^n : x_k \geq 0, \sum p_kx_k \leq B\} = \frac{1}{n!}\frac{B^n}{\prod_{k=1}^n p_k}.$$

For any two vectors $\boldsymbol{f}_1, \boldsymbol{f}_2 \in \mathbb{R}^n$ and two real numbers a, b consider a set $\{\boldsymbol{x}\in\mathbb{R}^n : x_k \geq 0, (\boldsymbol{f}_1,\boldsymbol{x}) = a,\ (\boldsymbol{f}_2,\boldsymbol{x}) = b\}$. Let $S_{\boldsymbol{f}_1,\boldsymbol{f}_2}(a,b)$ denotes the $n-2$-dimensional Lebesgue volume of this set.

Lemma 7.

The following equality holds:

$$S_{\mathbf{1}^n,\boldsymbol{p}^n}(A,B) = F_{n,\boldsymbol{p}}(A,B)\sqrt{n(p_1^2+\cdots+p_n^2)-(p_1+\cdots+p_n)^2}. \tag{14}$$

Proof: Consider a plane, spanned by the vectors $\mathbf{1}^n = (1, \dots, 1)$ and $\boldsymbol{p}^n = (p_1, \dots, p_n)$. Let $\boldsymbol{e}_1, \boldsymbol{e}_2$ be any orthonormal basis in this plane. Applying the Fubini theorem, for any domain $\Omega \subset \mathbb{R}^2$, we have

$$\mathrm{Vol}\{\boldsymbol{x} \in \mathbb{R}^n : x_k \geq 0, ((\boldsymbol{x}, \boldsymbol{e}_1), (\boldsymbol{x}, \boldsymbol{e}_2)) \in \Omega\} = \int\int_\Omega S_{\boldsymbol{e}_1, \boldsymbol{e}_2}(\xi, \eta) d\xi d\eta \,.$$

On the other hand, by the two-dimensional Newton-Leibnitz formula, for any $\Omega' \subset \mathbb{R}^2$ we have

$$\mathrm{Vol}\{\boldsymbol{x} \in \mathbb{R}^n : x_k \geq 0, ((\boldsymbol{x}, \mathbf{1}^n), (\boldsymbol{x}, \boldsymbol{p}^n)) \in \Omega'\} = \int\int_{\Omega'} F_{n,\boldsymbol{p}}(A, B) dA dB \,.$$

Suppose the domain Ω in the ξ, η-variables corresponds to the domain Ω' in A, B-variables; then we have

$$\int\int_\Omega S_{\boldsymbol{e}_1, \boldsymbol{e}_2}(\xi, \eta) d\xi d\eta = \int\int_{\Omega'} F_{n,\boldsymbol{p}}(A, B) dA dB.$$

Making the change of variables

$$\int\int_\Omega S_{\boldsymbol{e}_1, \boldsymbol{e}_2}(\xi, \eta) d\xi d\eta = \int\int_{\Omega'} S_{\mathbf{1}^n, \boldsymbol{p}^n}(A, B) \frac{\partial(\xi, \eta)}{\partial(A, B)} dA dB \,,$$

we obtain

$$\int\int_{\Omega'} S_{\mathbf{1}^n, \boldsymbol{p}^n}(A, B) \frac{\partial(\xi, \eta)}{\partial(A, B)} dA dB = \int\int_{\Omega'} F_{n,\boldsymbol{p}}(A, B) dA dB$$

for any domain Ω'. It remains to note that the Jacobean $\frac{\partial(\xi,\eta)}{\partial(A,B)}$ of the a linear map $(\xi, \eta) = (\xi(A, B), \eta(A, B))$ is equal to fraction of the area of the unit square in the ξ, η-variables ($= 1$) and the unit square in the A, B-variables. The last is equal to the area of the parallelogram spanned by the vectors $\mathbf{1}^n$, $\boldsymbol{p}^n$

$$= |\mathbf{1}^n||\boldsymbol{p}^n| \sin \angle(\mathbf{1}^n, \boldsymbol{p}^n) = \sqrt{n(p_1^2 + \cdots + p_n^2) - (p_1 + \cdots + p_n)^2}.$$

Since the domain Ω' is arbitrary, the lemma is proven. □

Proposition 8. $F_{n,\boldsymbol{p}}(A, B) = A^{n-2} F_{n,\boldsymbol{p}}(1, \frac{B}{A})$.

Proof: The function $V(\cdot, \cdot)$ is homogeneous of degree n. Hence its second derivative is homogeneous of degree $n - 2$. □

Proposition 9. *The function $y \mapsto F_{n,\boldsymbol{p}}(1, y)$ is nonnegative, with the support*

$$\mathrm{supp}\, F_{n,\boldsymbol{p}}(1, \cdot) = [\min_{k=1,\dots,n} p_k \,,\ \max_{k=1,\dots,n} p_k],$$

and

$$\int_{-\infty}^{\infty} F_{n,\boldsymbol{p}}(1, y)dy = \frac{1}{(n-1)!}.$$

If all p_k are positive; then

$$\int_{0}^{\infty} F_{n,\boldsymbol{p}}(x, 1)dx = \frac{1}{(n-1)!}\frac{1}{\prod_{k=1}^{n} p_k}.$$

Proof: The statement about the support is obvious. Calculate the integrals. Using the two-dimensional Newton-Leibnitz formula, we have

$$\frac{1}{n!} = \operatorname{vol}\{\boldsymbol{x} \in \mathbb{R}^n : x_k \geq 0\,,\ \sum x_k \leq 1\} = \int_0^1 \int_{-\infty}^{\infty} F_{n,\boldsymbol{p}}(x, y)dydx.$$

Using the homogeneity property, we obtain

$$\int_0^1 \int_{-\infty}^{\infty} F_{n,\boldsymbol{p}}(x, y)dydx = \int_0^1 \int_{-\infty}^{\infty} x^{n-2} F_{n,\boldsymbol{p}}(1, \frac{y}{x})dydx$$
$$= \int_0^1 \int_{-\infty}^{\infty} x^{n-1} F_{n,\boldsymbol{p}}(1, \eta)d\eta dx = \frac{1}{n}\int_{-\infty}^{\infty} F_{n,\boldsymbol{p}}(1, \eta)d\eta.$$

This proves the first equality. The second can be proved similarly starting from the following formula:

$$\frac{1}{n!}\frac{1}{\prod\limits_{k=1}^{n} p_k} = \operatorname{vol}\{\boldsymbol{x} \in \mathbb{R}^n : x_k \geq 0, \sum p_k x_k \leq 1\} = \int_0^1 \int_{-\infty}^{\infty} F_{n,\boldsymbol{p}}(x, y)dxdy. \ \square$$

Notation. Let $n_q = \#\{k \in \{1, \ldots, n\} : p_k = q\}$ be the multiplicity of the value q.

We note that $\sum_q n_q = n$. To help the reader to get used with this notation we remark that

$$\prod_{\substack{j=1,\ldots,n\\ p_j \neq q}} (p_j - \xi) = \prod_{q' \in \{p_1,\ldots,p_n\}\setminus q} (q' - \xi)^{n_{q'}}.$$

Both the left hand side and the right hand side contain $(n - n_q)$-many factors.

Notation. Define functions $(\cdot)_+^m$ and $(\cdot)_-^m$ as follows

$$(x)_+^m = \begin{cases} x^m, & x \geq 0, \\ 0, & x < 0, \end{cases} \qquad (x)_-^m = \begin{cases} 0, & x \geq 0, \\ x^m, & x < 0. \end{cases}$$

we note that $(x)_+^m + (x)_-^m \equiv x^m$.

Theorem 10. *The function $F_{n,\boldsymbol{p}}(1,\cdot)$ admits the representation:*

$$F_{n,\boldsymbol{p}}(1,c)=\frac{1}{(n-2)!}\sum_{q\in\{p_k\}_{k=1}^n}\frac{(-1)^{n_q-1}}{(n_q-1)!}\frac{d^{n_q-1}}{d\xi^{n_q-1}}\left(\frac{((c-\xi)_+)^{n-2}}{\prod\limits_{\substack{j=1,\dots,n\\ p_j\neq q}}(p_j-\xi)}\right)\Bigg|_{\xi=q}. \tag{15}$$

Equivalently, we have

$$F_{n,\boldsymbol{p}}(1,c)=\frac{1}{(n-2)!}\sum_{\substack{q\in\{p_k\}_{k=1}^n\\ q<c}}\frac{(-1)^{n_q-1}}{(n_q-1)!}\frac{d^{n_q-1}}{d\xi^{n_q-1}}\left(\frac{(c-\xi)^{n-2}}{\prod\limits_{\substack{j=1,\dots,n\\ p_j\neq q}}(p_j-\xi)}\right)\Bigg|_{\xi=q}. \tag{16}$$

Proof: In the particular case if all multiplicities are only the ones (and the zeros), i.e., all p_k are different we need to proof the following formula

$$F_{n,\boldsymbol{p}}(1,c)=\frac{1}{(n-2)!}\sum_{k=1}^{n}\frac{((c-p_k)_+)^{n-2}}{\prod\limits_{\substack{j=1,\dots,n\\ j\neq k}}(p_j-p_k)}. \tag{17}$$

Without loss of the generality we can restrict ourselves to this case. This is because the case with multiplicities can be obtained by the limiting procedure using the formula

$$\lim_{\varepsilon\to 0}\sum_{s=0}^{l-1}(-1)^s\varepsilon^{1-l}\frac{\varphi(x+s\varepsilon)}{s!(l-s-1)!}=\frac{(-1)^{l-1}}{(l-1)!}\varphi^{(l-1)}(x). \tag{18}$$

The proof of (17) is based on geometrical properties of $F_{n,\boldsymbol{p}}(1,c)$ (Lemma 7).

Consider linear functions:

$$l_{\mathbf{1}}(x_1,\dots,x_n)=x_1+\cdots+x_n;\quad l_{\boldsymbol{p}}(x_1,\dots,x_n)=p_1x_1+\cdots+p_nx_n.$$

We now proving formula (17). Without loss of generality we assume that $\{p_k\}$ are ordered: $p_1<\cdots<p_n$. Consider a simplex $\boldsymbol{T}$, constructed by the intersection of the positive octant of $\mathbb{R}^n$ and the hyper-plane $x_1+\cdots+x_n=1$. (The simplex $\boldsymbol{T}$ is the standard $(n-1)$-dimensional simplex. Its volume is: $\mathrm{vol}^{(n-1)}(\boldsymbol{T})=\frac{\sqrt{n}}{(n-1)!}$.) We denote the vertexes of $\boldsymbol{T}$ by $P_1,\dots,P_n$ in the order of increasing of the function $l_{\boldsymbol{p}}$. In particular, this means that $l_{\boldsymbol{p}}(P_k)=p_k$. This ordering is called natural. The n vertexes of the simplex $\boldsymbol{T}$ define $\frac{n(n-1)}{2}$ lines. For each of these lines we choose orientation, i.e., choose the direction, according with the increasing of the function $l_p(\cdot)$. For each vertex P_k of the simplex $\boldsymbol{T}$ we assign a positive $(n-1)$-dimensional octant O_k in the hyperplane $l_{\mathbf{1}}(\cdot)=1$. This octant constructed as follows. Given the vertex P_k, consider $n-1$ lines contained P_k and one of each other vertexes. Since each line is oriented, the positive ray with P_k as origin is defined (the ray is

directed according with the increasing of the function $l_{\boldsymbol{p}}(\cdot)$). Define O_k to be the convex hull of these $n-1$ rays.

Now, denoting the algebraic (i.e., counting the multiplicity) operation of addition and subtraction of the sets by $+$ and $-$ we have

$$\boldsymbol{T} = O_1 - O_2 + O_3 - O_4 + \cdots + (-1)^{n-1} O_n. \tag{19}$$

For each $k = 1, \ldots, n$ consider the following functions

$$\varphi_k(c) = \mathrm{vol}^{\mathrm{n-1}}(O_k \cap \{l_{\boldsymbol{p}}(\cdot) \le c\}),$$
$$\psi_k(c) = \mathrm{vol}^{\mathrm{n-2}}(O_k \cap \{l_{\boldsymbol{p}}(\cdot) = c\}).$$

We remark that the these quantities are finite, since the sets $O_k \cap \{l_{\boldsymbol{p}}(\cdot) \le c\}$ and $O_k \cap \{l_{\boldsymbol{p}}(\cdot) = c\}$ are bounded (by the construction of the sets O_k). Furthermore, we note that $\varphi_k(c)$ and $\psi_k(c)$ are equal to zero for $c \le p_k$.

In view of (19) and (14) we have

$$F_{n,\boldsymbol{p}}(1, c) = \frac{\sum_{k=1}^{n} (-1)^{k-1} \psi_k(c)}{\sqrt{n\left(p_1^2 + \cdots + p_n^2\right) - (p_1 + \cdots + p_n)^2}}. \tag{20}$$

Moreover, we have

$$\varphi_k(c) = \frac{1}{n-1} \psi_k(c) \,\mathrm{dist}\,(P_k, \{l_{\mathbf{1}} = 1, l_{\boldsymbol{p}} = c\}), \tag{21}$$

and

$$\mathrm{dist}(P_k, \{l_{\mathbf{1}} = 1, l_{\boldsymbol{p}} = c\}) = \frac{|c - p_k|}{\sqrt{\left(p_1^2 + \cdots + p_n^2\right) - \frac{1}{n}(p_1 + \cdots + p_n)^2}}. \tag{22}$$

Now, to calculate $\psi_k(\cdot)$, we need to find $\varphi_k(c)$. Assume $c > p_k$ (otherwise $\varphi_k(c) = 0$). The line defined by the vertexes $P_k\ P_j$, $j \ne k$ can be parameterized as follows:

$$t \mapsto P_k + t(P_j - P_k).$$

Define t_{kj} from the following equality

$$l_{\boldsymbol{p}}(P_k + t_{kj}(P_j - P_k)) = c \Leftrightarrow p_k + t_{kj}(p_j - p_k) = c \quad \Rightarrow \quad t_{kj} = \frac{c - p_k}{p_j - p_k}.$$

With chosen parameterization value $t = 0$ corresponds to the vertex P_k, and the value $t = 1$ corresponds to the vertex P_j. Consider a point on the line (P_k, P_j), for which $l_p = c$. The distance between this point and P_k is $\left|t_{kj}\right| \mathrm{dist}(P_k, P_j)$. Hence we have

$$\varphi_k(c) = \mathrm{vol}^{\mathrm{n}-1}(\boldsymbol{T}) \prod_{\substack{j=1 \\ j\neq k}}^{n} |t_{kj}| \quad \Rightarrow \quad (-1)^{k-1}\varphi_k(c) = \frac{\sqrt{n}}{(n-1)!} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{(c-p_k)_+}{p_j - p_k}. \tag{23}$$

Now can find $\psi_k(c)$ using (21), (22) and (23). Substituting the expression of $\psi_k(c)$ into (20), we arrive at (17). The theorem is proven. □

4. RANDOM VARIABLE x_m

Notation. Let $\boldsymbol{p} = (p_1, \dots, p_n)$ be a vector of the dimension n. Then $\boldsymbol{p}'_m$ be the vector with $n-1$ components, obtained by "discarding" the p_m $(1 \le m \le n)$.

Notation. For $x \in [0, 1)$ we set:

$$\varrho_{c,n,m}(x) = \frac{F_{n-1,\boldsymbol{p}'_m}(1-x, c-p_m x)}{F_{n,\boldsymbol{p}}(1,c)} = \frac{(1-x)^{n-3} F_{n-1,\boldsymbol{p}'_m}(1, \frac{c-p_m x}{1-x})}{F_{n,\boldsymbol{p}}(1,c)}. \tag{24}$$

For $x \notin [0, 1)$ we put $\varrho_{c,n,m}(x) = 0$. The function $\varrho_{c,n,m}$ is the (marginal) density of the distribution of the random variable x_m where the random variable $(x_1, \dots, x_n)$ uniformly distributed in the polyhedron

$$\sum x_j = 1, \quad \sum p_j x_j = c, \quad x_j \ge 0.$$

This follows from lemma 7 (saying that the value of $\varrho_{c,n,m}(x)$ is proportional to the measure of the corresponding section of the simplex) and the following proposition about normalization:

Proposition 11. *We have* $\displaystyle\int_0^1 \varrho_{c,n,m}(x)dx = 1.$

Proof: First we prove a simple fact.

Let L be an ℓ-dimensional hyperplane in $\mathbb{R}^n$, ($\ell \le n$), let $S \subset L$ be a bounded set. Fix any vector $\boldsymbol{l} \in \mathbb{R}^n$. Let $S_\xi = S \cap \{\boldsymbol{x} \in \mathbb{R}^n : (\boldsymbol{x}, \boldsymbol{l}) = \xi\}$. Then we have

$$\mathrm{vol}^{\ell}(S) = \frac{1}{\varkappa} \int_{-\infty}^{+\infty} \mathrm{vol}^{\ell-1}(S_\xi) d\xi,$$

where $\varkappa$ is the length of the (orthogonal) projection of $\boldsymbol{l}$ onto L.

Take L to be $\{\boldsymbol{x} \in \mathbb{R}^n : (\boldsymbol{1}^n, \boldsymbol{x}) = 1\,, (\boldsymbol{p}^n, \boldsymbol{x}) = c\}$. Take S to be the intersection of L and the positive octant of $\mathbb{R}^n$ Take $\boldsymbol{l}$ to be the m-th unit basis vector of the space

$\mathbb{R}^n$. Lemma 7 yields

$$\mathrm{vol}^{\mathrm{n}-2}(S) = \kappa_1\, F_{n,\boldsymbol{p}}(1, c),$$

$$\mathrm{vol}^{\mathrm{n}-3}(S_\xi) = \begin{cases} \kappa_2\, F_{n-1,\boldsymbol{p}'_m}(1-\xi, c - p_m\xi), & \text{if} \quad \xi \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Here we have used the notation:

$$\kappa_1 = \sqrt{n(p_1^2 + \cdots + p_n^2) - (p_1 + \cdots + p_n)^2},$$

$$\kappa_2 = \sqrt{(n-1)\sum_{k\neq m} p_k^2 - (\sum_{k\neq m} p_k)^2}.$$

The value of $\varkappa$ is the length of the projection of the $m^{\text{-th}}$ unit basis vector onto the hyperplane L. This hyperplane is orthogonal to the vectors $\mathbf{1}^n$ and $\boldsymbol{p}^n$. We note that the vector $\boldsymbol{p}^n$ can be replaced with $\bar{\boldsymbol{p}}^n$ where $\bar{p}_k = p_k - \frac{1}{n}\sum p_j$. Now we can find $\varkappa$ using the Pythagorean theorem:

$$\varkappa^2 = 1 - \frac{1}{n} - \frac{\left(p_m - \frac{\sum_{k=1}^n p_k}{n}\right)^2}{\sum_{j=1}^n \left(p_j - \frac{\sum_{k=1}^n p_k}{n}\right)^2}.$$

It remains to note that $\varkappa = \kappa_2/\kappa_1$. □

Corollary 11. *Consider the particular case $c = p_m$. From (23) and proposition (11) we have*

$$1^\circ\ \varrho_{p_m,n,m}(x) = (n-2)(1-x)^{(n-3)}, \quad 2^\circ\ \int_0^1 x^i \varrho_{p_m,n,m}(x)dx = \frac{i!(n-2)!}{(i+n-2)!}$$

Calculation 12. Assume that the first term p_1 is minimal and separated away from other terms of the sequence $\{p_k\}_{k=1}^\infty$. In other words, there exists $\sigma > p_1$ such that for any $k \geq 2$ we have $\sigma \leq p_k$. Then for any real number $c \in [p_1, \sigma]$,

any integer $m \geq 2$ and any integer $n \geq m$ we have

$$\int_0^\infty x^i \varrho_{c,n,m}(x)dx = \left(\frac{c - p_1}{p_m - p_1}\right)^i \frac{i!(n-2)!}{(i+n-2)!}.$$

Proof: From (23) we have

$$\varrho_{c,n,m}(x) = \begin{cases} 0, & \text{for } x > (c-p_1)/(p_m-p_1) \text{ or } x < 0, \\ (n-2)\left(1 - \frac{p_m-p_1}{c-p_1}x\right)^{n-3} \frac{p_m-p_1}{c-p_1}, & \text{otherwise.} \end{cases}$$

Integrating this, we obtain :

$$\int_0^\infty x^i \varrho_{c,n,m}(x)dx = \int_0^{\frac{c-p_1}{p_m-p_1}} x^i (n-2)\left(1 - \frac{p_m - p_1}{c - p_1}x\right)^{n-3}$$

$$\frac{p_m - p_1}{c - p_1}dx = (n-2)\left(\frac{c-p_1}{p_m-p_1}\right)^i \int_0^1 z^i(1-z)^{n-3}dz$$

$$= (n-2)\left(\frac{c-p_1}{p_m-p_1}\right)^i \frac{i!(n-3)!}{(i+n-2)!}.$$

□

Calculation 13. Assume that all terms of the sequence $\{p_k\}_{k=1}^n$ are different. Let $1 \leq m \leq n$ and $\min_{k=1,\ldots,n} p_k \leq c \leq \max_{k=1,\ldots,n} p_k$. Then for $c \leq p_m$ we have

$$\int_0^1 x^i \varrho_{c,n,m}(x)dx = \frac{\displaystyle\sum_{k=1,\ldots,n\,:\,p_k<c} \frac{(c-p_k)^{n-2}}{\prod_{\substack{j=1\\j\neq k}}^n (p_j - p_k)}\left(\frac{c-p_k}{p_m-p_k}\right)^i}{\displaystyle\sum_{k=1,\ldots,n\,:\,p_k<c} \frac{(c-p_k)^{n-2}}{\prod_{\substack{j=1\\j\neq k}}^n (p_j - p_k)}} \frac{i!(n-2)!}{(i+n-2)!}, \tag{25}$$

and for $c \geq p_m$ we have

$$\int_0^1 x^i \varrho_{c,n,m}(x)dx = \frac{\displaystyle\sum_{k=1,\ldots,n\,:\,p_k>c} \frac{(c-p_k)^{n-2}}{\prod_{\substack{j=1\\j\neq k}}^n (p_j - p_k)}\left(\frac{c-p_k}{p_m-p_k}\right)^i}{\displaystyle\sum_{k=1,\ldots,n\,:\,p_k>c} \frac{(c-p_k)^{n-2}}{\prod_{\substack{j=1\\j\neq k}}^n (p_j - p_k)}} \frac{i!(n-2)!}{(i+n-2)!}. \tag{26}$$

Proof: Consider the case $c \leq p_m$. Without loss of generality we assume that $p_1, \ldots, p_s$ are less than c, and $p_{s+1}, \ldots, p_n$ are greater than or equal to c.

By virtue of (17) and (24) we have

$$\varrho_{c,n,m}(x) = \frac{\displaystyle(n-2)\sum_{k=1}^{s}(p_m - p_k)\frac{((c-p_k-(p_m-p_k)x)_+^{n-3}}{\prod_{\substack{j=1,\ldots,n\\j\neq k}} (p_j - p_k)}}{\displaystyle\sum_{k=1}^{s} \frac{(c-p_k)^{n-2}}{\prod_{\substack{j=1,\ldots,n\\j\neq k}} (p_j - p_k)}}. \tag{27}$$

Both the numerator denominator contains only s terms, because other terms (for $s > k$) are equal to zero.

We note that for $p_k < c < p_m$ we have

$$(n-2)(p_m - p_k)\int_0^1 x^i (c - p_k - (p_m - p_k)x)_+^{n-3} dx =$$

$$(c - p_k)^{n-2} \left(\frac{c - p_k}{p_m - p_k}\right)^i \frac{i!(n-2)!}{(i+n-2)!}.$$

This completes the proof for the case $c \le p_m$.

In the case $c \ge p_m$ we use the representation

$$F_{n,\boldsymbol{p}}(1, c) = \frac{-1}{(n-2)!} \sum_{k=1}^{n} \frac{((c - p_k)_-)^{n-2}}{\prod\limits_{\substack{j=1,\dots,n \\ j \ne k}} (p_j - p_k)} \tag{28}$$

and repeat our arguments.

Formula (28) follows from (17) and the polynomial in c identity:

$$\frac{1}{(n-2)!} \sum_{k=1}^{n} \frac{(c - p_k)^{n-2}}{\prod\limits_{\substack{j=1,\dots,n \\ j \ne k}} (p_j - p_k)} \equiv 0.$$

This polynomial identity follows from the fact that $F_{n,\boldsymbol{p}}(1, c) \equiv 0$ for $c > \max p_k$. □

Remark. For $c \le p_m$ (respectively, for $c \ge p_m$) we can allow the multiplicities in the sequence $\{p_k\}$ for $p_k \ge c$ (respectively, for $p_k \le c$).

Calculation 14. Let $p_1 = p_2 = 1$ and $p_k \ge 2$ for $k > 2$ (as, e.g., for the half of the integer lattice). Then for $1 \le c \le 2$, $n > 2$ and $2 < m \le n$ we have

$$\int_0^1 x^i \varrho_{c,n,m}(x) dx = \frac{\left.\dfrac{d}{d\xi}\left(\dfrac{(c-\xi)^{n-2}}{\prod_{j=3}^n (p_j - \xi)} \left(\dfrac{c - \xi}{p_m - \xi}\right)^i\right)\right|_{\xi=1}}{\left.\dfrac{d}{d\xi}\left(\dfrac{(c-\xi)^{n-2}}{\prod_{j=3}^n (p_j - \xi)}\right)\right|_{\xi=1}} \frac{i!(n-2)!}{(i+n-2)!}.$$

Proof: This formula follows by a limiting procedure from the previous calculation and (16). □

Using the formula $f' = f(\log f)'$ we obtain the following proposition:

Calculation 15. Under the assumption of calculation (14) we have

$$\int_0^1 x^i \varrho_{c,n,m}(x)dx = \left(\frac{c-1}{p_m-1}\right)^i \frac{i!(n-2)!}{(i+n-2)!} R_{c,n,m},$$

where $R_{c,n,m}$ is defined as:

$$R_{c,n,m} = \frac{\frac{n+i-2}{c-1} - \sum_{j=3}^{n} \frac{1}{p_j-1} - \frac{i}{p_m-1}}{\frac{n-2}{c-1} - \sum_{j=3}^{n} \frac{1}{p_j-1}}.$$

For fixed c and i we have the following uniform convergence

$$R_{c,n,m} \xrightarrow[n\to\infty]{\text{uniformly in } m} 1\,.$$

Proposition 16. *Assume that the sequence $\{p_k\}$ and a number c satisfy the following property: There exists positive number ε such that only finite number of terms of the sequence $\{p_k\}$ less then $c+\varepsilon$. Assume that $p_1 \le p_k$ for all k. Then*

$$n^i(p_m - p_1)^i \int_0^1 x^i \rho_{c,n,m}(x)dx \xrightarrow[n\to\infty]{\text{uniformly in } m:p_m>c} i!(c-p_1)^i.$$

Proof: Denote

$$W_q^{(s)} = \frac{(-1)^s}{s!}\frac{d^s}{d\xi^s}\left(\frac{(c-\xi)^{n-2}}{\prod_{j=1,\dots,n:p_j\neq q}(p_j-\xi)}\left(\frac{c-\xi}{p_m-\xi}\right)^i\right)\Bigg|_{\xi=q},$$

similarly, we denote:

$$w_q^{(s)} = \frac{(-1)^s}{s!}\frac{d^s}{d\xi^s}\left(\frac{(c-\xi)^{n-2}}{\prod_{j=1,\dots,n:p_j\neq q}(p_j-\xi)}\right)\Bigg|_{\xi=q}.$$

The quantities $W_q^{(s)}$ and $w_q^{(s)}$ also depend on c, n, m and the sequence $\{p_k\}$, however we do not reflect it in the notation to avoid cumbersome notation. Using (23) and (18), we have:

$$\int_0^1 x^i \rho_{c,n,m}(x)dx = \frac{\sum_{q\in\cup_{p_k<c}\{p_k\}} W_q^{(n_q-1)}}{\sum_{q\in\cup_{p_k<c}\{p_k\}} w_q^{(n_q-1)}}\frac{i!(n-2)!}{(i+n-2)!}. \tag{29}$$

The proof follows from the fact that for $q_1 < q_2$ we have

$$\lim_{n\to\infty} \frac{W_{q_2}^{(n_{q_2}-1)}}{W_{q_1}^{(n_{q_1}-1)}} = 0 \qquad \lim_{n\to\infty} \frac{w_{q_2}^{(n_{q_2}-1)}}{w_{q_1}^{(n_{q_1}-1)}} = 0.$$

This means that only the first terms contribute to the limit. Moreover, we have

$$\lim_{n\to\infty} \frac{W_q^{(n_q-1)}}{w_q^{(n_q-1)}} = \left(\frac{c-q}{p_m-q}\right)^i.$$

Due to the comment above, we need this property only for $q = p_1$, The proposition is proven. □

We now will prove that "for not too big" c the quantity $\int_0^1 x\varrho_{c,n,m}(x)dx$ can be approximated by $\frac{c-p_1}{(m-p_1)(n-1)}$.

Proposition 17. *Suppose $p_k = k$; then for each $\varepsilon > 0$ there exist $N > 1$ and $K > 1$, such that $\forall n > N, \forall m \in [2, \ldots, n]$, and for any*

$$c \in \left(1, \min\left\{m, \frac{n}{K\log n}\right\}\right)$$

we have

$$\left|\int_0^1 x\varrho_{c,n,m}(x)dx \frac{(m-1)(n-1)}{c-1} - 1\right| < \frac{\varepsilon}{n}.$$

Proof: Without loss of generality we can assume that $\varepsilon < 1/3$.

Since $c < m$, calculation 13 give us the following expression (cf. (29)):

$$\int_0^1 x\varrho_{c,n,m}(x)dx = \frac{W_1 + W_2 + W_3 + \cdots + W_{[c]}}{w_1 + w_2 + w_3 + \cdots + w_{[c]}} \frac{1}{n-1},$$

where

$$W_k = (-1)^{k-1}\frac{(c-k)^{n-2}}{(k-1)!(n-k)!}\frac{c-k}{m-k}, \qquad w_k = (-1)^{k-1}\frac{(c-k)^{n-2}}{(k-1)!(n-k)!}.$$

The main point of the proof is that if $c < \frac{n}{K\log n}$ then the terms $W_2, W_3, \ldots$ and $w_2, w_3, \ldots$ are much smaller than W_1 and w_1 respectively, for large n. Besides we have

$$\frac{W_1}{w_1} = \frac{c-1}{m-1}. \tag{30}$$

Remark. If p_1 is a multiple value then we need to replace (30) with $\lim_{n\to\infty} \frac{W_1}{w_1} = \frac{c-p_1}{m-p_1}$. The rate of convergence can be estimated from above as $\frac{const}{n}$ (cf. calculation 15).

We have

$$\frac{W_2}{W_1} > \frac{W_3}{W_2} > \frac{W_4}{W_3} > \cdots \quad \text{and} \quad \frac{w_2}{w_1} > \frac{w_3}{w_2} > \frac{w_4}{w_3} > \cdots .$$

Moreover, the condition $c < m$ implies

$$\frac{W_2}{W_1} < \frac{w_2}{w_1}.$$

Hence it is sufficient to prove that $\frac{w_2}{w_1}$ is "very small". In other words we will choose N and K such that for

$$n > N \quad \text{and} \quad c < \min\left\{m, \frac{n}{K \log n}\right\}$$

we have

$$\frac{w_2}{w_1} < \frac{\varepsilon}{10n}.$$

This is equivalent to the following inequality

$$(n-1)\left(\frac{c-2}{c-1}\right)^{n-2} < \frac{\varepsilon}{10n}.$$

We see that it is sufficient to take K and N such that the condition

$$c < \frac{n}{K \log n} \qquad \text{for } n > N$$

implies the inequality

$$\left(\frac{c-2}{c-1}\right)^{n-2} < \frac{\varepsilon}{10n^2}. \tag{31}$$

In this case the quantity $\frac{W_1+W_2+W_3+\cdots}{w_1+w_2+w_3+\cdots}$ is "almost" the quantity $\frac{W_1}{w_1}$, and (30) concludes the proof.

Inequality (31) is equivalent to the following relation

$$(n-2)\log\left(\frac{c-2}{c-1}\right) < \log(\varepsilon/10) - 2\log n.$$

Since $\log(\frac{c-2}{c-1}) < -\frac{1}{c-1}$, then (31) follows from any of following equivalent inequalities:

$$-\frac{n-2}{c-1} < \log(\varepsilon/10) - 2\log n \quad \Leftrightarrow \quad c < \frac{n-2}{2\log n - \log(\varepsilon/10)} + 1\,.$$

This means that we can take $K = 3$ and $N = N(\varepsilon)$. The proposition is proven. $\square$

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